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EFFICIENT RAMSEY EQUILIBRIA

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Ramsey equilibrium models with heterogeneous agents and borrowing constraints are shown to yield efficient equilibrium sequences of aggregate capital and consumption. The proof of this result is based on verifying that equilibrium sequences of prices satisfy the Malinvaud criterion for efficiency.

Keywords: Ramsey Equilibria, Borrowing Constraint, Efficiency, Malinvaud Condition

1. INTRODUCTION

A fundamental question in macrodynamic models of capital accumulation concerns whether or not the economy is providing as much consumption as it can following a competitive equilibrium path. For optimal growth models, or their equivalent perfect foresight competitive economy counterparts, the answer is affirmative. The optimal program of capital accumulation invests neither too much, nor too little, over time.

In a seminal paper, Malinvaud (1953) found sufficient conditions for identifying efficient programs.[1] His theorem was designed to work within a wide range of model specifications, including models not yet developed when he wrote on this matter in the early 1950s. Since that time, representative agent and heterogenous agent models of capital accumulation with infinitely lived households endowed with perfect foresight over the future paths of prices, absent technological uncertainty or idiosyncratic risks, have been developed by a number of economic theorists.[2]

For models where the equilibrium program may not necessarily solve a social welfare problem, it is interesting to learn if the resulting path of capital accumulation is efficient and society is providing as much consumption as possible, even if that consumption is not necessarily achieving a Pareto optimal distribution. One class of these models, a form of the many-agent Ramsey model, consider

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heterogeneous infinitely lived agents with different rates of discounting future utility in a one-sector perfect foresight model. This framework, inspired by Ramsey (1928), can be found in a series of papers following the formulation, and proof, of the existence of a unique stationary equilibrium in which only the most patient household owns capital in Becker (1980). The key structural assumption in Becker's formulation is that households are forbidden to borrow against their anticipated future wage income. This borrowing constraint makes the model one with incomplete markets in a certainty setting. It is not reasonable to expect a form of the first welfare theorem to obtain. However, demonstrating that the resulting equilibrium is efficient is a minimal welfare test.

Malinvaud's sufficiency theorem highlights the way in which the efficiency criterion focuses solely on aggregate consumption, and not on how it is distributed to individuals. Yet we will see that how private individuals actually value their marginal consumption at each time plays a fundamental role in detecting whether or not an equilibrium is, in fact, giving rise to an efficient allocation of society's scarce capital and providing the most consumption possible in the aggregate.

Previous literature on efficiency in incomplete markets addressed this question in stochastic overlapping-generations models as well as in models of infinitely lived consumers operating in exchange economies with goods defined by their dates of availability and as state-contingent claims. The paper by Bloise and Reichlin (2009) that inspired the present work is a good example of this previous literature.[3]

Our paper's main result is that the Ramsey-equilibrium aggregate capital sequences are efficient provided that the most patient household's capital stock is eventually positive, and remains so thereafter. This condition, satisfied in all currently known examples, is sufficient to identify (eventually) that agent's subjective prices and the market prices. This agent's subjective prices obey a transversality condition that is transmitted to the marketplace; Malinvaud's theorem implies efficiency.

2. THE MALINVAUD CRITERION FOR EFFICIENCY

Production takes place using a single capital good. The productive technology turns labor and capital goods into a composite good that can be either consumed or saved as next period's capital input. The amount of labor is fixed in this economy (there will be one unit of labor services per household and all labor services are assumed to be identical). The technology is summarized by a *production function*, denoted by f. Let $y = f(k)$ denote the composite good y produced from a fixed amount of labor (whose value is suppressed in the notation), together with a nonnegative capital input k. Capital is assumed to depreciate completely within the period. Hence, the model is formally one with circulating capital that is consumed within the production period. The output y is available for consumption or capital accumulation with a one-period lag. The formal properties of f are recorded as Assumption I:

Assumption I. $f : \mathbf{R}_+ \to \mathbf{R}_+$, $f(0) = 0$, f is $C^{(2)}$ on $\mathbf{R}_{++}$, $f' > 0$, $\lim_{x\to 0} f'(k) = \infty$, $\lim_{x\to\infty} f'(k) = 0$, and $f'' < 0$.

This assumption implies there is a maximum sustainable capital stock, denoted B, satisfying $B = f(B) > 0$. Let $\mathbf{R}_+ = [0, \infty)$ and $\mathbf{R}_{++} = (0, \infty)$.

The capital stock sequence $\{K_{t-1}\}, t = 1, 2 \ldots$ is a *capital stock program* if $K_{t-1} \geq 0$ and $f(K_{t-1}) - K_t \geq 0$ for each $t \geq 1$. The *corresponding consumption program* is $\{C_t\}$ with $C_t = f(K_{t-1}) - K_t$. The capital stock program and corresponding consumption programs are *feasible* if $K_0 = \mathbf{k} > 0$, where $\mathbf{k}$ is the given starting stock. Assumption I implies that if the initial aggregate capital stock $\mathbf{k}$ is smaller than B, then all nonnegative sequences of consumption and capital satisfying the *balance condition*, $C_t + K_t = f(K_{t-1})$ for all t with $K_0 = \mathbf{k}$, are bounded from above by B.

A feasible capital stock program $\{K'_t\}$ *dominates* the feasible capital stock program $\{K_t\}$, with $K_t \neq K'_t$ for some t, if the corresponding consumption program, $\{C'_t\}$ has the property $C'_t \geq C_t$ for all t, with strict inequality for some t. A feasible capital stock program that is dominated is called *inefficient*; otherwise, it is said to be *efficient*.

Associated to any feasible capital program $\{K_t\}$, where $K_t > 0$ for all $t \geq 1$, is a sequence of *shadow prices* $\{p_t\}$, or *competitive prices*, which are recursively defined by

$$p_0 = 1,\; p_{t+1} f'(K_t) = p_t, \quad t \geq 0. \qquad \textbf{(P)}$$

These prices are also the ones *implied by* or *derived from* $\{K_t\}$. Note that such a price sequence has the property (given that f is concave)

$$p_{t+1} f(K_t) - p_t K_t \geq p_{t+1} f(x) - p_t x \text{ for each } x \geq 0 \text{ and each } t \geq 0. \qquad \textbf{(1)}$$

This is the *periodwise (or intertemporal) profit-maximizing condition*. The prices defined in this manner are strictly positive as $K_t > 0$ for each t.

In general, a sequence $\{K_t, p_t\}$ is *intertemporal profit–maximizing* if $\{K_t\}$ is a feasible capital program starting from $k_0 > 0$, $\{p_t\}$ is a nonnull, nonnegative price sequence, and (1) obtains for each $t \geq 0$.

Starting with Malinvaud (1953) many authors have shown a close connection between shadow prices and ascertaining whether or not the underlying feasible program is efficient.[4] *The Malinvaud Sufficiency Theorem* [Malinvaud (1953)] is

THEOREM 1. *Assume f satisfies AI. If a sequence $\{K_t, p_t\}$ is intertemporal profit–maximizing, with $p_t > 0$ for each $t \geq 0$, and*

$$\lim_{t\to\infty} p_t K_t = 0, \qquad \textbf{(2)}$$

then $\{K_t\}$ is efficient.

It is sufficient to verify $p_t \to 0$ as $t \to \infty$ for the models appearing in this paper.

Application of Malinvaud's theorem requires calculating the shadow prices. This is readily done for the case of the one-sector discounted Ramsey model of optimal growth. Well-known necessary and sufficient conditions include satisfaction of a transversality condition in the form (2) and that the optimum $\{K_{t-1}\}$ is efficient. Of course, this result is obvious, because an optimal capital sequence is under consideration; this previews our arguments.

Shadow prices satisfying (P) implicitly define a capital goods rental rate in each period by letting $f'(K_t) = 1 + r_{t+1}$. We turn this around and impute the market price sequence $\{p_t\}$ for an equilibrium from this profit condition and (P), with $p_0 = 1$ defining the numeraire.

3. THE RAMSEY-EQUILIBRIUM MODEL

Agents preferences assume time–additively separable utility functions with fixed discount factors. The technology is specified by a one-sector model with a single all-purpose consumption–capital good as before.

The general complete-market competitive one-sector model treats budget constraints as restricting the present value of an agent's consumption to be smaller than or equal to the agent's initial wealth, defined as the capitalized wage income plus the present value of that person's initial capital. This allows us to interpret the choice of a consumption stream as if the agent is allowed to borrow and lend at market-determined present-value prices, subject to repaying all loans. Markets are complete—any intertemporal trade satisfying the present-value budget constraint is admissible at the individual level. The Ramsey equilibrium model changes the budget constraint from a single one (reckoned as a present value) to a sequence, one for each period. Agents are forbidden to borrow against their future labor income, so they cannot capitalize the future wage stream into a present value. Markets are incomplete; individuals are debt-constrained. The operation of a borrowing constraint in the individual household problems also breaks the possibility of an equilibrium allocation arising as the economy's Pareto optimal allocation.

3.1. The Basic Model and Blanket Assumptions

There are $H \geq 1$ households, indexed by $h = 1, \ldots, H$. There is a single commodity available for consumption or investment at each time. At time zero, households are endowed with capital stocks $k^h \geq 0$. Put $\mathbf{k} = \sum_h k^h$ and assume $\mathbf{k} > 0$. Let c_t^h, x_t^h denote the consumption and capital stock of household h at time t. Household h has *felicity function* u_h; c_t^h is the argument of u_h. Household h discounts future utilities by the factor δ_h with $0 < \delta_h < 1$. Hence, the household's lifetime utility function is specified by $\sum_{t=1}^{\infty} \delta_h^{t-1} u_h(c_t^h)$.

Assumption II. For each h, $u_h : \mathbf{R}_+ \to \mathbf{R}$ is $C^{(2)}$ on $\mathbf{R}_{++}$ with $u_h' > 0,\ u_h'' < 0$, and $\lim_{c \to 0} u'(c) = \infty$.

We focus on the case where the first household's discount factor is larger than all the other households' discount factors. Assumption III orders households from the most patient to the least patient.

Assumption III. $1 > \delta_1 > \delta_2 \geq \cdots \geq \delta_H > 0$.

Production takes place using a single capital good as set out in Section 2. Assumptions I–III are blanket assumptions assumed for the remainder of this article and sometimes referred to as (AI)–(AIII). If $H = 1$, then the Ramsey equilibrium model coincides with the standard optimal growth problem. Assume $H \geq 2$ in the sequel.

3.2. The Households' Problems

Let $\{1 + r_t, w_t\}$ be a sequence of one-period rental factors and wage rates, respectively. The sequences $\{1+r_t, w_t\}$ are always taken to be nonnegative and nonzero. Households are competitive agents and perfectly anticipate the profile of factor returns $\{1 + r_t, w_t\}$. Given $\{1 + r_t, w_t\}$, h solves

$$P(h):\ \sup \sum_{t=1}^{\infty} \delta_h^{t-1} u_h\left(c_t^h\right)$$

by choice of nonnegative sequences $\{c_t^h, x_t^h\}$ satisfying $x_0^h = k^h$ and

$$c_t^h + x_t^h = w_t + (1 + r_t)x_{t-1}^h \quad (t = 1, 2, \ldots). \tag{3}$$

The market structure of this model requires capital assets to be nonnegative at each moment of time and requires that agents without capital cannot borrow against the discounted value of their future wage income.

The *no-arbitrage* or *Euler* necessary conditions for $\{c_t^h, x_t^h\}$ to solve $P(h)$ are $c_t^h > 0$ and

$$\delta_h(1 + r_{t+1})u_h'\left(c_{t+1}^h\right) \leq u_h'\left(c_t^h\right). \tag{4}$$

If $x_t^h > 0$, then the inequality in (4) can be reversed, resulting in the *Euler equation*:

$$\delta_h(1 + r_{t+1})u_h'\left(c_{t+1}^h\right) = u_h'\left(c_t^h\right). \tag{5}$$

The corresponding *transversality condition* is

$$\lim_{t\to\infty} \delta_h^{t-1} u_h'\left(c_t^h\right) = 0, \tag{6}$$

which also implies $\lim_{t\to\infty} \delta_h^{t-1} u_h'(c_t^h)x_{t-1}^h = 0$, because $\{x_{t-1}^h\}$ is a bounded sequence.

3.3. The Production Sector's Objective

The production sector is characterized by the one-sector neoclassical production function f satisfying Assumption I.

All the intertemporal decisions are taken in the household sector. Producers are supposed to take the rental rate as given and solve the myopic profit-maximization problem

$$P(F): \ \sup[f(x_{t-1}) - (1+r_t)x_{t-1}]$$

at each t by choice of $x_{t-1} \geq 0$. The residual profit is treated as the wage bill. It is shared equally by the identical households as wages—production is worker-owned.

If $0 < 1 + r_t < \infty$, then (AI) implies that there is a unique positive stock K_{t-1} that solves *P(F)* at each t. Clearly

$$f'(K_{t-1}) = 1 + r_t; \tag{7}$$

furthermore, the corresponding $\{w_t\}$ is positive, as defined by

$$Hw_t = f(K_{t-1}) - (1+r_t)K_{t-1}. \tag{8}$$

3.4. The Ramsey Economy and Its Equilibrium Concept

A collection $\mathcal{E} = (f, \{u_h, \delta_h, k^h\}, h = 1, 2, \ldots, H)$ satisfying Assumptions I–III, and for which $k^h \geq 0$ for each h with $\mathbf{k} = \sum_{h=1}^{H} k^h > 0$, $\mathbf{k} \leq B$, is said to be *a Ramsey economy*, or simply, *an economy*. The economy always has a positive aggregate capital stock, and at least one agent will always possess some capital at time zero.

The equilibrium concept is *perfect foresight*. Households perfectly anticipate the sequences of rental and wage rates. They solve their optimization problems for their planned consumption demand and capital supply sequences. The production sector calculates the capital demand at each time and the corresponding total output supply. Rentals are paid to the households for capital supplied and the residual profits are paid out as the total wage bill. An equilibrium occurs when the households' capital supply equals the production sector's capital demand at every point of time. A form of Walras's law implies that the total consumption demand and supply of capital for the next period equals current output. Thus, in equilibrium, every agent is maximizing its objective function and planned supply equals planned demand in every market.

DEFINITION 2. *Sequences* $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ *constitute a Ramsey equilibrium for a given economy* $\mathcal{E}$ *provided*

(E1) *For each h,* $\{c_t^h, x_{t-1}^h\}$*solves P(h) given* $\{1 + r_t, w_t\}$.
(E2) *For each t,* K_{t-1} *solves P(F) given* $1 + r_t$.
(E3) $Hw_t = f(K_{t-1}) - (1 + r_t)K_{t-1}$ $(t = 1, 2, \ldots)$.
(E4) $\sum_{h=1}^{H} x_{t-1}^h = K_{t-1}$ $(t = 1, 2, \ldots)$, $0 < \mathbf{k} = K_0 \leq B$.

The output market balance follows by combining (E1)–(E4):

$$\sum_{h=1}^{H}\left(c_t^h + x_t^h\right) = f(K_{t-1}). \tag{9}$$

Note that equilibrium consumption and capital sequences are bounded from above by the maximum sustainable stock. The assumed conditions for households and the producer imply that in an equilibrium $c_t^h > 0$ and $K_{t-1} > 0$ for each t, given that $\mathbf{k}$ is positive, and each agent's income $w_t + (1 + r_t)x_{t-1}^h > 0$ at each time, even if $x_{t-1}^h = 0$. Moreover, at least one household's capital stock is positive at each time along an equilibrium profile.

Given an equilibrium path, the corresponding *aggregate capital sequence and consumption sequence* are defined by the formulas $K_t = \sum_{h=1}^{H} x_t^h$ and $C_t = \sum_{h=1}^{H} c_t^h$, respectively. The Malinvaud criterion for testing efficiency is applied to these sequences.

4. PROPERTIES OF RAMSEY EQUILIBRIA

A Ramsey-equilibrium program is *stationary* for the economy $\mathcal{E}$ provided the equilibrium wage rate, the rental rate, the aggregate capital stock, and the allocations of capital and consumption are constant over time. Becker (1980) proves the existence of a unique stationary equilibrium in which only the most patient household has capital—all other households have none and live off their wage incomes.

Let K^{δ_1} be the unique solution to the equation $f'(k) = (1/\delta_1)$. This capital stock is the first household's capital and the *stationary aggregate capital stock* in the stationary equilibrium solution. *Stationary aggregate consumption* is found at each time by adding the economy's wage bill to the rental income received by the most patient household.

General properties of equilibrium paths found under Assumptions I–III are briefly summarized next. Formal details and proofs are in the referenced papers [e.g., see Becker and Foias (1987)]. Fix the economy $\mathcal{E}$ meeting Assumptions (I-III).

(P1) Equilibria exist.[5]

(P2) If $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ is a Ramsey equilibrium for $\mathcal{E}$, then the no-capital state is recurrent for each $h \geq 2$. That is, for each $h \geq 2$, $x_t^h = 0$ infinitely often.

This recurrence theorem is the most general result in the literature on the properties enjoyed in a dynamic Ramsey equilibrium. It tells us that households $h \geq 2$ achieve the zero-capital state infinitely often. There are equilibria where agents more impatient than the first hold capital infinitely often. See Stern's example in Becker (2006).

The *turnpike property* obtains if every $h \geq 2$ eventually reaches a no-capital position and maintains that state thereafter. Stern's example implies the turnpike property does not generally obtain. Yet it holds whenever each household $h \geq 2$ is sufficiently myopic in comparison to the first household's discount factor.[6] It also obtains whenever the equilibrium aggregate capital stock sequence is convergent and that limit must be the steady state stock.[7]

(P3) For each equilibrium, $\limsup_{t\to\infty} K_{t-1} \geq K^{\delta_1}$.

This result does not exclude the capital sequence from exceeding the Golden-Rule capital stock, K^g, infinitely often, where K^g is defined as the solution to $f'(k) = 1$. This is important, as this situation could be a way for a path to be inefficient. Cass (1972) notes that a periodic path could be efficient or inefficient if it oscillated around the Golden–Rule stock.[8] The two-period cycles found by Becker and Foias (1987) and Stern [see Becker (2006)] oscillate around the Golden-rule Stock, so they are potential counterexamples to the general efficiency of Ramsey-equilibrium programs. Period-two equilibrium cycles are shown later to be efficient.

(P4) Each household's consumption is bounded away from zero along an equilibrium path. That is, $\eta^h \equiv \inf_t c_t^h > 0$ $(h = 1, 2, \ldots, H)$ holds in each equilibrium.[9]

Property (P4) implies that no agent consumes zero or even approaches zero consumption asymptotically. This result distinguishes the Ramsey model with borrowing constraints from its complete-market general-equilibrium counterparts.[10] This bound is the critical property used to show that the appropriate sequence of supporting prices satisfies the transversality condition sufficient for efficiency.

5. THE EFFICIENCY OF A RAMSEY-EQUILIBRIUM PROGRAM

The definition of an efficient capital stock sequence is applied to the *aggregate capital stock sequence*, $\{K_{t-1}\}$, in an economy $\mathcal{E}$ given the equilibrium $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$. In this case, $K_{t-1} = \sum_{h=1}^H x_{t-1}^h$ and the corresponding *aggregate consumption* is the sequence $\{C_t\}$ with $C_t = \sum_{h=1}^H c_t^h$. The paths $\{K_{t-1}, C_t\}$ are feasible from the initial stocks $\mathbf{k}$ (the distribution of initial capital across households does not enter the discussion). For the purposes of efficiency analysis, the question is whether the aggregate capital stock sequence $\{K_{t-1}\}$ is dominated by another feasible aggregate capital sequence $\{K_{t-1}^*\}$ with its corresponding aggregate consumption $\{C_t^*\}$ defined periodwise by $C_t^* = f(K_{t-1}^*) - K_{t-1}^*$, $K_0^* = \mathbf{k}$. The test of whether $\{K_{t-1}\}$ is dominated by a particular $\{K_{t-1}^*\}$ places no restrictions on how C_t^* is allocated to the individual households at any time t.

5.1. Efficient Programs: Two Examples

Efficiency of the equilibrium $\{K_{t-1}\}$ can be verified directly in some cases where a priori qualitative or quantitative information about the equilibrium aggregate capital sequence is known.

Example 1: A monotone increasing capital stock sequence

Becker and Foias (1987) show that if AI–AIII and the *capital-income monotonicity condition* holds, then the sequence $\{K_{t-1}\}$ is eventually monotonic and converges to K^{δ_1} as $t \to \infty$. Capital-income monotonicity holds if $f'(k)k$ is an increasing function of k; it is satisfied if $f(k) = Ak^{\alpha}$ for some $A > 0$ and $0 < \alpha < 1$.

Consider the general case where $K_t \to K^{\delta_1}$. Because $f'(K^{\delta_1}) = (1/\delta_1) > 1$, and $\theta \equiv [1 + (1/\delta_1)]/2$ satisfies $1 < \theta < (1/\delta_1)$, concavity of f on $\mathbf{R}_+$ and continuity of f' on $\mathbf{R}_{++}$ imply that there is a positive integer T such that for all $t \geq T$, we have $f'(K_t) \geq \theta > 1$. Thus $\{p_t\}$ defined previously satisfies $p_t \to 0$ as $t \to \infty$, and $\{K_t\}$ is efficient by Malinvaud's sufficiency theorem.

Example 2: A two-period equilibrium capital stock sequence

Periodic-equilibrium capital sequences present challenges for demonstrating that the aggregate capital sequence is efficient whenever they oscillate around the Golden-Rule capital stock. The examples of two-period Ramsey equilibria found in Becker and Foias (1987) and Stern, as published in Becker (2006), oscillate around the Golden-Rule stock. Becker and Foias assume that only agent 1 has capital. Stern's example has the second household holding capital infinitely often; the first household always has capital.

Becker and Foias's example has the following piecewise linear production function structure:

$$f(k) = \begin{cases} 10 + 5k & \text{for } 0 \leq k \leq 10 \\ 52 + (4/5)k & \text{for } k \geq 10. \end{cases}$$

This example, and the ones developed by Stern [see Becker (2006) and Sorger (1994; 1995)], fail the capital income monotonicity test (otherwise, the paths would be convergent).[11]

Let $K_0 = 12 := K_H$ and let $K_L = 8$, with $f'(8) = 5$ and $f'(12) = (4/5)$. Note that the Golden-Rule stock occurs at $k = 10$, where we note that 1 is a supergradient of f at $K^g = 10$. The path $\{12, 8, 12, 8, \ldots\}$ can be shown to be an equilibrium two-cycle capital sequence for appropriate choices of the discount factors and utility functions for the two households. Compute p_t to find

$$p_t = \begin{cases} 1/4^{(t/2)} & \text{if } t \text{ is an even number} \\ 1.25/4^{(t-1)/2} & \text{if } t \text{ is an odd number.} \end{cases}$$

Here, $p_0 = 1$. Observe that the sequence $(p_t) \to 0$; this implies that the equilibrium prices in this period-two capital sequence are efficient, by Malinvaud's theorem. It turns out that two-period equilibria are always efficient (see Section 5.3).

5.2. The Efficiency Theorem

The previous examples have one common feature: the definition of an appropriate system of shadow prices to check the Malinvaud Sufficiency Theorem is readily available from the detailed knowledge of the equilibrium aggregate capital sequence. It is known that other equilibrium dynamics for the aggregate capital sequence are possible than being monotonic or cycling with period 2. The goal of this section is to offer a general sufficient condition to detect efficiency of a Ramsey-equilibrium capital sequence: *the first household eventually has a positive capital stock and maintains a positive stock for all subsequent times*. All known examples of Ramsey-equilibria satisfy this hypothesis. This condition is weaker than those implying the turnpike property.

The No-Arbitrage Inequality (4) may be rewritten for each h in a given equilibrium as

$$\delta_h u_h'\left(c_{t+1}^h\right)/u_h'\left(c_t^h\right) \leq (1/f'(K_t)) \text{ for each } t \geq 1. \tag{10}$$

The left-hand side of (10) is *h's subjective intertemporal discount factor* for consumption in period $t+1$ when viewed at time t. The right-hand side is the corresponding *market discount factor* (the reciprocal of the market interest factor). The inequality (10) is a necessary condition for optimality for this household. Moreover, if $x_t^h > 0$, then (10) is an equality.

Define agent h's *subjective present-value consumption price at time t* by the formula

$$p_t^h := \delta_h^{t-1} u_h'\left(c_t^h\right); \text{ with } p_0^h \equiv 1. \tag{11}$$

Using this definition, and rewriting (10), yields

$$\left(p_{t+1}^h/p_t^h\right) \leq (1/f'(K_t)) = (p_{t+1}/p_t) \tag{12}$$

for each $t \geq 1$ and with equality if $x_t^h > 0$. Along an equilibrium path some agent always has positive capital, so (12) holds as an equality for some agent at each time.

LEMMA 3. *Make Assumptions I–III. Let $\{1+r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ be a Ramsey equilibrium for an economy $\mathcal{E}$. Then $\{p_t^1 = \delta_1^{t-1} u_1'(c_t^1)\}$, with $p_0^1 \equiv 1$, satisfies*

$$\sum_{t=0}^{\infty} p_t^1 < \infty \tag{13}$$

and therefore the transversality condition holds:

$$\lim_{t\to\infty} p_t^1 = 0. \tag{14}$$

Proof. The strict concavity of u_1 and $\eta^1 = \inf_t c_t^1 > 0$ [by (P4)] imply for the given equilibrium path that

$$0 < u_1'\left(c_t^1\right) \leq u_1'(\eta^1) < \infty; \tag{15}$$

(13) and (14) follow. ■

This prepares us for the main *efficiency theorem*:

THEOREM 4. *Make Assumptions I–III. Let* $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ *be a Ramsey equilibrium for an economy* $\mathcal{E}$. *Suppose there is some positive integer* T, *such that for each* $t \geq T$, $x_t^1 > 0$. *Then*

$$\sum_{t=0}^{\infty} p_t < \infty \tag{16}$$

holds and the equilibrium program's capital stock sequence is efficient.

Proof. Using (12), we obtain, for all $t \geq T$,

$$\left(p_{t+1}^1/p_t^1\right) \leq [1/f'(K_t)] = (p_{t+1}/p_t). \tag{17}$$

This yields (by iteration on (17)) for all $t \geq T$,

$$\left(p_{t+1}^1/p_T^1\right) = (p_{t+1}/p_T). \tag{18}$$

Because (13) holds by Lemma 3, (18) implies that (16) must obtain. Thus, we have $p_t \to 0$ as $t \to \infty$, and $\{K_t\}$ is efficient by Malinvaud's Sufficiency Theorem. ■

5.3. Applications of the Efficiency Theorem

Several applications illustrate the efficiency theorem.

Multiple, periodic, and chaotic equilibria. Sorger (1994), Theorem 1, proved it is possible for an economy $\mathcal{E}$ satisfying (AI)–(AIII), given the equilibrium $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ with the fixed initial condition **k**, to exhibit multiple equilibria from the same initial conditions. He showed that there are economies for which there is a stationary equilibrium with $k^1 = K^{\delta_1} = \mathbf{k}$, $k^h = 0$, and another equilibrium from the same initial distribution of the capital stock **k** having period p, where p is a natural number, $p \geq 3$. That is, there are two equilibrium programs from the same initial distribution of capital. In his constructed Ramsey equilibria, the most patient household always holds the entire capital stock, and therefore the efficiency theorem implies that *both equilibria are efficient*. Similarly, the chaotic equilibria found by Sorger (1995) are also efficient. The latter paths cannot be computed exactly, but the efficiency theorem guarantees that the resulting aggregate capital sequences are efficient.

Two cycles are efficient. A Ramsey equilibrium $\{(1 + r_t), w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ is a *period-two Ramsey-equilibrium cycle* if there exist $\hat{x}$ and $\bar{x}$ in $\mathbf{R}_+^H$ with $\hat{x} \neq \bar{x}$, such that

$$x_t \equiv \left(x_t^1, \ldots, x_t^H\right) = \begin{cases} \hat{x} & \text{for } t = 0, 2, 4, \ldots \\ \bar{x} & \text{for } t = 1, 3, 5, \ldots \end{cases}. \tag{19}$$

PROPOSITION 5. *Let* $\{(1+r_t), w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ *be a period-two Ramsey-equilibrium cycle. Then*

(a) $x_t^1 > 0$ *for all* $t \geq 0$, *and*
(b) *the Ramsey equilibrium is efficient.*

Proof. Because $\{(1+r_t), w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ is a period-two Ramsey-equilibrium cycle, there exist $\hat{x}$ and $\bar{x}$ in $\mathbf{R}_+^H$ with $\hat{x} \neq \bar{x}$, such that (19) holds. Define $\hat{K} = \sum_{h=1}^H \hat{x}^h$ and $\bar{K} = \sum_{h=1}^H \bar{x}^h$. Then we have $K_t = \hat{K}$ for $t \in \{0, 2, 4, \ldots\}$, and $K_t = \bar{K}$ for $t \in \{1, 3, 5, \ldots\}$.Without loss of generality, let $\bar{K} = \max\{\hat{K}, \bar{K}\}$. (Note that $\hat{K} = \bar{K}$ is not ruled out.) Then, by (P3), we have $\bar{K} = \lim_{t\to\infty} \sup K_t \geq K^{\delta_1}$, and therefore

$$\delta_1 f'(\bar{K}) \leq \delta_1 f'(K^{\delta_1}) = 1. \tag{20}$$

To establish (a), we analyze two cases separately. We have either (i) $\bar{x}^1 = 0$, or (ii) $\bar{x}^1 > 0$.

Case (i): In this case, there is some $h \in \{2, \ldots, H\}$, such that $\bar{x}^h > 0$. Without loss of generality, denote this h by 2. Then $\bar{x}^2 > 0$ and by (P2), $\hat{x}^2 = 0$. Pick any $T \in \{1, 3, 5, \ldots\}$. Then $K_T = \bar{K}$ and $x_T^2 = \bar{x}^2 > 0$, whereas $x_{T+1}^2 = x_{T-1}^2 = \hat{x}^2 = 0$. Thus, we get

$$\begin{aligned} &(i)\ c_T^2 = w_T + (1+r_T)x_{T-1}^2 - x_T^2 = w_T - x_T^2 < w_T \\ &(ii)\ c_{T+1}^2 = w_{T+1} + (1+r_{T+1})x_T^2 - x_{T+1}^2 > w_{T+1}. \end{aligned} \tag{21}$$

Further, $K_T = \bar{K} = \max\{\bar{K}, \hat{K}\} \geq \hat{K} = K_{T-1}$, so that

$$w_{T+1} = [f(K_T) - K_T f'(K_T)]/H \geq [f(K_{T-1}) - K_{T-1} f'(K_{T-1})]/H = w_T. \tag{22}$$

Thus, (21) and (22) imply $c_{T+1}^2 > c_T^2$. Since $x_T^2 > 0$, the Ramsey–Euler equation holds, and yields $1 > u_2'(c_{T+1}^2)/u_2'(c_T^2) = [1/\delta_2 f'(K_T)]$. Thus, we get

$$\delta_2 f'(K_T) > 1. \tag{23}$$

But, using (20), we have $\delta_2 f'(K_T) < \delta_1 f'(K_T) = \delta_1 f'(\bar{K}) \leq \delta_1 f'(K^{\delta_1}) = 1$, which contradicts (23). Thus case (i) cannot arise.

Case (ii): In this case, we have $\bar{x}^1 > 0$. We claim now that $\hat{x}^1 > 0$. If the claim were not true, then $\hat{x}^1 = 0$. Pick any $T \in \{1, 3, 5, \ldots\}$. Then $K_T = \bar{K}$ and $x_T^1 = \bar{x}^1 > 0$, whereas $x_{T-1}^1 = x_{T+1}^1 = \hat{x}_1 = 0$. Now, following steps (21) and (22), replacing household 2 by household 1, we get

$$\delta_1 f'(K_T) > 1. \tag{24}$$

But, by using (20), we have: $\delta_1 f'(K_T) = \delta_1 f'(\bar{K}) \leq \delta_1 f'(K^{\delta_1}) = 1$, which contradicts (24). This establishes our claim. Thus, $x_t^1 > 0$ for all $t \geq 0$, proving part (a) of the proposition.

Part (b) follows directly from part (a) and Theorem 4. ■

Maximum consumption value. An interesting corollary follows from the efficiency theorem. The sequence of aggregated consumption defined by the given Ramsey-equilibrium path, $\{(\sum_{h=1}^{H} c_t^h)\}$, is bounded (from above, by the maximum sustainable stock, B). Hence,

$$\sum_{t=1}^{\infty} p_t \left(\sum_{h=1}^{H} c_t^h \right) < \infty \qquad \textbf{(25)}$$

holds as well and the conditions are met to apply a result obtained by Cass and Yaari (1971) to conclude the following about the *maximum value of aggregate consumption* in a Ramsey equilibrium:

COROLLARY 6. *Make Assumptions I–III. Let* $\{1+r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ *be a Ramsey equilibrium for an economy* $\mathcal{E}$*. Suppose, in addition, for this equilibrium, there is a time* $T < \infty$ *such that* $t \geq T$ *implies* $x_t^1 > 0$*. Then, for any feasible consumption program* $\{c_t\}$ *starting from the same initial stocks* $\boldsymbol{k}$*,*

$$\sum_{t=1}^{\infty} p_t c_t \leq \sum_{t=1}^{\infty} p_t \left(\sum_{h=1}^{H} c_t^h \right),$$

where $\{p_t\}$ *is defined by (P). That is, the present discounted value of aggregate consumption is maximized in a Ramsey equilibrium calculated at the system of shadow prices* $\{p_t\}$*.*

Proof. The theorem and corollary of Cass and Yaari (1971) apply to yield the conclusion because the given Ramsey equilibrium is efficient. ■

This corollary answers the basic question posed in the Introduction. It gives a precise sense in which society achieves as much consumption as possible from its economic system. Here, the maximum consumption possibility is measured by the discounted value of the equilibrium aggregate consumption stream.

The efficiency theorem and its corollary focuses on aggregate consumption and capital accumulation. The marginal valuations in the shadow prices reflect the private consumption values of agents holding capital, who implicitly have the largest willingness to pay for a marginal unit of the composite consumption–capital good at each time. Their marginal valuations agree with the market's valuation, which reflects capital's marginal productivity at each time. The sufficiency condition ensures that the long-run foresight of the most patient agent is reflected in the price system. The invisible hand promotes the economy's efficient allocation of its scarce capital as the most patient agent pursues its self-interest.

6. CONCLUSION

The proof that Ramsey equilibria are efficient relied on an auxilliary assumption on equilibrium sequences rather than conditions solely placed on the model's economic primitives governing tastes, endowments, and technology. One open problem is to verify that *all* Ramsey equilibria are efficient. Any candidate for an *inefficient equilibrium* would necessarily require that the first household enter a zero-capital state infinitely often.

It is perhaps a surprise that many borrowing-constrained Ramsey equilibrium models allocate society's scarce capital efficiently. But this says nothing about how the economy's consumption is actually distributed across agents. A major remaining problem is to examine the model for second-best or constrained Pareto optima that reflect the limitations on intertemporal exchange derived from the borrowing constraints.

NOTES

1. The broader search for a complete characterization of efficient programs, at least in one-sector models, was resolved by Cass (1972).The references include citations to key works that generalized and extended the Cass efficiency criterion following the publication of Cass (1972).
2. This paper's bibliography includes many such selections.
3. Also, see Alvarez and Jermann (2000), Bloise (2008), Bloise and Calciano (2008), Chattopadhyay and Gottardi (1999) and Chattopadhyay (2008) for studies on efficiency in stochastic models with various forms of market incompleteness, including default and borrowing constraints.
4. See Benveniste and Gale (1975), Cass (1972), and Mitra (1979) for general criteria for efficiency in one-sector models.
5. See Becker et al. (1991) for general existence theorems that apply to the additive separable utility cases in this paper, as well as for broader recursive utility specifications.
6. See Becker and Tsyganov's paper [Lemma 4.4, Becker and Tsyganov (2002)]. Their result is derived for a two-sector model, but applies to one-sector models by assuming that both sectors have indentical production functions.
7. See Becker and Foias [Propositions 4 and 5 in Becker and Foias (1987)] for a proof.
8. See Cass (1972, p. 214).
9. A formal proof of this fact is available on request from Robert Becker as a Technical Appendix.
10. See Bewley (1982), Coles (1985), Le Van and Vailakis (2003), and Rader (1972).
11. Note that the piecewise linear functional form can be smoothed to satisfy the conditions necessary to invoke Malinvaud's theorem as well as to apply Ramsey-equilibrium theory.

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